CONVERGENCE OF THE ALLEN-CAHN EQUATION WITH CONSTRAINT TO BRAKKE'S MEAN CURVATURE FLOW

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Abstract. In this paper we consider the Allen-Cahn equation with constraint. In 1994, Chen and Elliott [7] studied the asymptotic behavior of the solution of the Allen-Cahn equation with constraint. They proved that the zero level set of the solution converges to the classical solution of the mean curvature flow under the suitable conditions on initial data. In 1993, Ilmanen [18] proved the existence of the mean curvature flow via the Allen-Cahn equation without constraint in the sense of Brakke. We proved the same conclusion for the Allen-Cahn equation with constraint.

1. Introduction

Let $T > 0$ and $\varepsilon \in (0,1)$. In this paper, we consider the following Allen-Cahn equation with constraint:

$$\begin{cases} \partial_t \varphi^\varepsilon - \Delta \varphi^\varepsilon + \dfrac{\partial I_{[-1,1]}(\varphi^\varepsilon) - \varphi^\varepsilon}{\varepsilon^2} \ni 0, & (x,t) \in \mathbb{R}^n \times (0,T), \\ \varphi^\varepsilon(x,0) = \varphi_0^\varepsilon(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.1}$$

Here, $I_{[-1,1]}$ is the indicator function of $[-1,1]$ defined by

$$I_{[-1,1]}(s) = \begin{cases} 0, & \text{if } s \in [-1,1], \\ +\infty, & \text{otherwise}, \end{cases}$$

and $\partial I_{[-1,1]}$ is the subdifferential of $I_{[-1,1]}$, that is

$$\partial I_{[-1,1]}(s) = \begin{cases} \emptyset, & \text{if } s < -1 \text{ or } s > 1, \\ [0,\infty), & \text{if } s = 1, \\ \{0\}, & \text{if } -1 < s < 1, \\ (-\infty, 0], & \text{if } s = -1. \end{cases}$$

Set

$$\mathcal{G} := \{ v \in L^\infty(\mathbb{R}^n) : \|v\|_{L^\infty(\mathbb{R}^n)} \leq 1 \} \quad \text{and} \quad \mathcal{K} := \mathcal{G} \cap H^1(\mathbb{R}^n).$$

For $\varphi_0^\varepsilon \in \mathcal{G}$, $\varphi^\varepsilon \in C(0,T;L^2(\mathbb{R}^n))$ is called a solution for (1.1) if the following hold:

$$\begin{cases} \varphi^\varepsilon \in L^2(0,T;H^1(\mathbb{R}^n)),\ \partial_t \varphi^\varepsilon \in L^2(0,T;(H^1(\mathbb{R}^n))'), \\ \varphi^\varepsilon(\cdot,t) \in \mathcal{K} \text{ a.e. } t \in (0,T),\ \varphi^\varepsilon(\cdot,0) = \varphi_0^\varepsilon(\cdot), \\ \int_0^T \{ \langle \partial_t \varphi^\varepsilon, v - \varphi^\varepsilon \rangle + (\nabla \varphi^\varepsilon, \nabla (v - \varphi^\varepsilon)) - \frac{1}{\varepsilon^2}(\varphi^\varepsilon, v - \varphi^\varepsilon) \} \, dt \geq 0 \\ \text{for any } v \in L^2(0,T;H^1(\mathbb{R}^n)), \quad \text{with} \quad v(\cdot,t) \in \mathcal{K} \text{ for any } t \in (0,T). \end{cases}$$

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Here $\langle\ ,\ \rangle$ denotes the pairing of $(H^1(\mathbb{R}^n))'$ and $H^1(\mathbb{R}^n)$, and $(\ ,\ )$ denotes the inner product in $L^2(\mathbb{R}^n)$.

Let $\delta \in (0, \frac{1}{2})$. To study (1.1), we consider the following equation:

$$\begin{cases} \partial_t \varphi^{\varepsilon,\delta} - \Delta \varphi^{\varepsilon,\delta} + \dfrac{F_\delta'(\varphi^{\varepsilon,\delta})}{\varepsilon^2} = 0, & (x,t) \in \mathbb{R}^n \times (0,\infty), \\ \varphi^{\varepsilon,\delta}(x,0) = \varphi_0^{\varepsilon,\delta}(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.2}$$

where

$$F_\delta(s) = \begin{cases} \dfrac{1-\delta}{2\delta}\Big(s + \dfrac{1}{1-\delta}\Big)^2, & \text{if } s < -1, \\ -\dfrac{1}{2}s^2 + \dfrac{1}{2(1-\delta)}, & \text{if } |s| \le 1, \\ \dfrac{1-\delta}{2\delta}\Big(s - \dfrac{1}{1-\delta}\Big)^2, & \text{if } s > 1. \end{cases}$$

The function $F_\delta'(s)$ is the Yosida approximation of $\partial I_{[-1,1]}(s) - s$. We remark that $F \in C^1(\mathbb{R})$, $F_\delta(s) \ge 0$ and $F_\delta(s) = 0$ if and only if $s = \pm(1-\delta)^{-1}$. By an argument similar to that in [7], the classical solution of (1.2) converges to the solution of (1.1) under the suitable conditions on initial data as $\delta \to 0$ for any $T > 0$.

The purpose of this paper is to prove that the solutions of (1.1) and (1.2) converge to a weak solution for the mean curvature flow. Here, a family of hypersurfaces $\{\Gamma(t)\}_{t\in[0,T)}$ is called the mean curvature flow if the velocity of $\Gamma(t)$ is

$$V_\Gamma = H \quad \text{on } \Gamma(t), \quad t \in (0,T), \tag{1.3}$$

where H is the mean curvature vector of $\Gamma(t)$. Chen and Elliott [7] proved that for a classical solution $\{\Gamma(t)\}_{t\in[0,T)}$ of the mean curvature flow, there exists a family of functions $\{\varphi_0^\varepsilon\}_{\varepsilon>0}$ such that the zero level set of the solution φ^ε for (1.1) converges to $\{\Gamma(t)\}_{t\in[0,T)}$ as $\varepsilon \to 0$. But there is no result for the construction of the global weak solution for the mean curvature flow via (1.1) or (1.2).

In this paper, we consider a weak solution for the mean curvature flow called Brakke's mean curvature flow which we define later [4]. There is a large amount of research on the mean curvature flow [3, 8, 11, 12, 13, 14] and the connection between the Allen-Cahn equation and the mean curvature flow [2, 5, 6, 10], so we may mention only a part of them related to Brakke's mean curvature flow and (1.1). Brakke [4] proved the existence of a Brakke's mean curvature flow by using geometric measure theory. Ilmanen [18] proved that the singular limit of the Allen-Cahn equation without constraint is a Brakke's mean curvature flow under mild conditions on initial data. The main results of this paper is the same conclusion for (1.1) and (1.2). Liu, Sato and Tonegawa [22], and Takasao and Tonegawa [25] proved that there exists Brakke's mean curvature flow with transport term via the phase field method. Moreover, the regularity of Brakke's mean curvature flow was proved by Kasai and Tonegawa [20] and Tonegawa [26] by improving on Brakke's partial regularity theorem for mean curvature flow. Recently, Farshbaf-Shaker, Fukao and Yamazaki [15] characterized the Lagrange multiplier λ^ε of (1.1), where $\lambda^\varepsilon = \lambda^\varepsilon(\varphi^\varepsilon)$ satisfies

$$(\partial_t \varphi^\varepsilon, \psi) + (\nabla\varphi^\varepsilon, \nabla\psi) + \frac{1}{\varepsilon^2}(\lambda^\varepsilon, \psi) = \frac{1}{\varepsilon^2}(\varphi^\varepsilon, \psi)$$

for any $\psi \in H^1(\mathbb{R}^n)$ and a.e. $t \in (0,T)$. Suzuki, Takasao and Yamazaki [24] studied the criteria for the standard forward Euler method to give stable numerical experiments of (1.1).

The organization of the paper is as follows. In Section 2 of this paper we set out the basic definitions and explain the main results. In Section 3 we study the monotonicity formula and prove some propositions. In Section 4 we show the existence of limit measure μ_t which

corresponds to $\Gamma(t)$. In Section 5 we prove the density lower bound of μ_t and the vanishing of the discrepancy measure ξ. In Section 6 we show the main results.

2. Preliminaries and main results

We recall some notations from geometric measure theory and refer to [1, 4, 9, 16, 23] for more details. On $\mathbb{R}^n$ we denote the Lebesgue measure by $\mathcal{L}^n$. Define $\omega_n := \mathcal{L}^n(B_1(0))$. For $r > 0$ and $a \in \mathbb{R}^n$ we define $B_r(a) := \{x \in \mathbb{R}^n \mid |x-a| < r\}$. We denote the space of bounded variation functions on $\mathbb{R}^n$ as $BV(\mathbb{R}^n)$. We write the characteristic function of a set $A \subset \mathbb{R}^n$ as χ_A. For a set $A \subset \mathbb{R}^n$ with finite perimeter, we denote the total variation measure of the distributional derivative $\nabla\chi_A$ by $\|\nabla\chi_A\|$. For $a = (a_1, a_2, \ldots, a_n)$, $b = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$ we denote $a \otimes b := (a_i b_j)$. For $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{n\times n}$, we define

$$A : B := \sum_{i,j=1}^{n} a_{ij} b_{ij}.$$

Let $G_k(\mathbb{R}^n)$ be the Grassman manifold of unoriented k-dimensional subspaces in $\mathbb{R}^n$. Let $S \in G_k(\mathbb{R}^n)$. We also use S to denote the n by n matrix representing the orthogonal projection $\mathbb{R}^n \to S$. Especially, if $k = n-1$ then the projection for $S \in G_{n-1}(\mathbb{R}^n)$ is given by $S = I - \nu \otimes \nu$, where I is the identity matrix and ν is the unit normal vector of S. Let $S^\perp \in G_{n-k}(\mathbb{R}^n)$ be the orthogonal complement of S.

We call a Radon measure on $\mathbb{R}^n \times G_k(\mathbb{R}^n)$ a general k-varifold in $\mathbb{R}^n$. We denote the set of all general k-varifolds by $\mathbf{V}_k(\mathbb{R}^n)$. Let $V \in \mathbf{V}_k(\mathbb{R}^n)$. We define a mass measure of V by

$$\|V\|(A) := V((\mathbb{R}^n \cap A) \times G_k(\mathbb{R}^n))$$

for any Borel set $A \subset \mathbb{R}^n$. We also denote

$$\|V\|(\phi) := \int_{\mathbb{R}^n \times G_k(\mathbb{R}^n)} \phi(x)\, dV(x,S) \quad \text{for} \quad \phi \in C_c(\mathbb{R}^n).$$

The first variation $\delta V : C_c^1(\mathbb{R}^n;\mathbb{R}^n) \to \mathbb{R}$ of $V \in \mathbf{V}_k(\mathbb{R}^n)$ is defined by

$$\delta V(g) := \int_{\mathbb{R}^n \times G_k(\mathbb{R}^n)} \nabla g(x) : S\, dV(x,S) \quad \text{for} \quad g \in C_c^1(\mathbb{R}^n;\mathbb{R}^n).$$

We define a total variation $\|\delta V\|$ to be the largest Borel regular measure on $\mathbb{R}^n$ determined by

$$\|\delta V\|(G) := \sup\{\delta V(g) \mid g \in C_c^1(G;\mathbb{R}^n),\ |g| \le 1\}$$

for any open set $G \subset \mathbb{R}^n$. If $\|\delta V\|$ is locally bounded and absolutely continuous with respect to $\|V\|$, by the Radon-Nikodym theorem, there exists a $\|V\|$-measurable function $H(x)$ with values in $\mathbb{R}^n$ such that

$$\delta V(g) = -\int_{\mathbb{R}^n} H(x) \cdot g(x)\, d\|V\|(x) \quad \text{for} \quad g \in C_c(\mathbb{R}^n;\mathbb{R}^n).$$

We call H the generalized mean curvature vector of V.

Let $\mathcal{H}^k$ be the k-dimensional Hausdorff measure. We call a Radon measure μ k-rectifiable if μ is represented by $\mu = \theta\mathcal{H}^k\lfloor M$, that is, $\mu(\phi) = \int_{\mathbb{R}} \phi\, d\mu = \int_M \phi\theta\, d\mathcal{H}^k$ for any $\phi \in C_c(\mathbb{R}^n)$. Here M is countably k-rectifiable and $\mathcal{H}^k$-measurable, and $\theta \in L^1_{loc}(\mathcal{H}^k\lfloor M)$ is positive valued $\mathcal{H}^k$-a.e. on M. For a k-rectifiable Radon measure $\mu = \theta\mathcal{H}^k\lfloor M$ we define a unique k-varifold V by

$$\int_{\mathbb{R}^n \times G_k(\mathbb{R}^n)} \phi(x,S)\, dV(x,S) := \int_{\mathbb{R}^n} \phi(x, T_x\mu)\, d\mu(x) \qquad \text{for } \phi \in C_c(\mathbb{R}^n \times G_k(\mathbb{R}^n)), \tag{2.1}$$

where $T_x\mu$ is the approximate tangent space of M at x. Note that $T_x\mu$ exists $\mathcal{H}^k$-a.e. on M in this assumption, and $\mu = \|V\|$ under this correspondence.

Definition 2.1. Let μ be a Radon measure on $\mathbb{R}^n$ and $\phi \in C_c^2(\mathbb{R}^n;\mathbb{R}^+)$. We define

$$\mathcal{B}(\mu,\phi) := \int_{\mathbb{R}^n} -\phi|H|^2 + \nabla\phi\cdot(T_x\mu)^\perp\cdot H\,d\mu$$

if $\mu\lfloor\{\phi>0\}$ is rectifiable, $\|\delta V\|\lfloor\{\phi>0\} \ll \mu\lfloor\{\phi>0\}$ and $\int_{\mathbb{R}^n}|H|^2\,d\mu<\infty$. Here V is a k-varifold defined by (2.1) and H is the the generalized mean curvature vector of V. If any one of the condition is not satisfied, then we define $\mathcal{B}(\mu,\phi) := -\infty$.

Definition 2.2. A family $\{\mu_t\}_{t\geq 0}$ of Radon measures is called Brakke's mean curvature flow if

$$\overline{D}_t\mu_t(\phi) \leq \mathcal{B}(\mu_t,\phi) \tag{2.2}$$

is hold for any $\phi\in C_c^2(\mathbb{R}^n;\mathbb{R}^+)$ and any $t\geq 0$. Here $\overline{D}f(t) = \overline{\lim}_{h\to 0}\frac{f(t+h)-f(t)}{h}$ is the upper derivative.

Definition 2.3. Let $\varphi^{\varepsilon,\delta}$ be a solution for (1.2). We define a Radon measure $\mu_t^{\varepsilon,\delta}$ by

$$\mu_t^{\varepsilon,\delta}(\phi) := \int_{\mathbb{R}^n}\phi\Big(\frac{\varepsilon|\nabla\varphi^{\varepsilon,\delta}|^2}{2} + \frac{F_\delta(\varphi^{\varepsilon,\delta})}{\varepsilon}\Big)dx, \tag{2.3}$$

for any $\phi\in C_c(\mathbb{R}^n)$.

For $r\in\mathbb{R}$ we define

$$q^\varepsilon(r) := \begin{cases} -1, & \text{if } r < -\frac{\varepsilon\pi}{2}, \\ \sin\frac{r}{\varepsilon}, & \text{if } |r|\leq\frac{\varepsilon\pi}{2}, \\ 1, & \text{if } r>\frac{\varepsilon\pi}{2} \end{cases} \tag{2.4}$$

and

$$q^{\varepsilon,\delta}(r) := \begin{cases} \frac{\delta}{1-\delta}e^{\sqrt{\frac{1-\delta}{\delta}}\sin^{-1}\sqrt{1-\delta}}e^{\frac{r}{\varepsilon}\sqrt{\frac{1-\delta}{\delta}}} - \frac{1}{1-\delta}, & \text{if } r<-\varepsilon\sin^{-1}\sqrt{1-\delta}, \\ \frac{1}{\sqrt{1-\delta}}\sin\frac{r}{\varepsilon}, & \text{if } |r|\leq\varepsilon\sin^{-1}\sqrt{1-\delta}, \\ -\frac{\delta}{1-\delta}e^{\sqrt{\frac{1-\delta}{\delta}}\sin^{-1}\sqrt{1-\delta}}e^{-\frac{r}{\varepsilon}\sqrt{\frac{1-\delta}{\delta}}} + \frac{1}{1-\delta}, & \text{if } r>\varepsilon\sin^{-1}\sqrt{1-\delta}. \end{cases} \tag{2.5}$$

Remark 2.4. (1) $q^\varepsilon\in C^{1,\alpha}(\mathbb{R})$, $q^{\varepsilon,\delta}\in C^2(\mathbb{R})$ and for any $\varepsilon>0$ we have

$$\lim_{\delta\to 0}\|q_0^{\varepsilon,\delta} - q_0^\varepsilon\|_{C^{1,\alpha}(\mathbb{R})} = 0. \tag{2.6}$$

(2) $q^{\varepsilon,\delta}$ is a solution for

$$\frac{\varepsilon(q_r^{\varepsilon,\delta})^2}{2} = \frac{F_\delta(q^{\varepsilon,\delta})}{\varepsilon} \qquad\text{and}\qquad q_{rr}^{\varepsilon,\delta} = \frac{F_\delta'(q^{\varepsilon,\delta})}{\varepsilon^2} \tag{2.7}$$

with $q^{\varepsilon,\delta}(0)=0$, $q^{\varepsilon,\delta}(\pm\infty) = \pm(1-\delta)^{-1}$, $q^{\varepsilon,\delta}(\pm\varepsilon\sin^{-1}\sqrt{1-\delta}) = \pm 1$ and $q_r^{\varepsilon,\delta}(r)>0$ for any $r\in\mathbb{R}$. Moreover we have

$$\sup_{r\in\mathbb{R},\delta\in(0,\frac12)}|q_r^{\varepsilon,\delta}(r)|\leq 2\varepsilon^{-1} \quad\text{and}\quad \sup_{r\in\mathbb{R},\delta\in(0,\frac12)}|q_{rr}^{\varepsilon,\delta}(r)|\leq 2\varepsilon^{-2}. \tag{2.8}$$

(3) By (2.7) we have

$$\begin{aligned}\int_{\mathbb{R}}\frac{\varepsilon(q_r^{\varepsilon,\delta})^2}{2} + \frac{F_\delta(q^{\varepsilon,\delta})}{\varepsilon}\,dr &= \int_{\mathbb{R}}\sqrt{2F_\delta(q^{\varepsilon,\delta})}q_r^{\varepsilon,\delta}\,dr \\ &= \int_{-(1-\delta)^{-1}}^{(1-\delta)^{-1}}\sqrt{2F_\delta(s)}\,ds =: \sigma_\delta.\end{aligned} \tag{2.9}$$

Let $\Omega_0^+ \subset \mathbb{R}^n$ be a bounded open set and we denote $\Gamma_0 := \partial\Omega_0^+$. Throughout this paper, we assume the following:

(1) There exists $D_0 > 0$ such that

$$\sup_{x\in\mathbb{R}^n, R>0} \frac{\mathcal{H}^{n-1}(\Gamma_0 \cap B_R(x))}{\omega_{n-1}R^{n-1}} \leq D_0 \quad \text{(Density upper bounds)}. \tag{2.10}$$

(2) There exists a family of open sets $\{\Omega_0^i\}_{i=1}^\infty$ such that Ω_0^i have a C^3 boundary Γ_0^i such that (Ω_0^+, Γ_0) be approximated strongly by $\{(\Omega_0^i, \Gamma_0^i)\}_{i=1}^\infty$, that is

$$\lim_{i\to\infty} \mathcal{L}^n(\Omega_0^+ \triangle \Omega_0^i) = 0 \quad \text{and} \quad \lim_{i\to\infty} \|\nabla\chi_{\Omega_0^i}\| = \|\nabla\chi_{\Omega_0^+}\| \ \text{ as measures.} \tag{2.11}$$

Remark 2.5. If Γ_0 is C^1, then (2.10) and (2.11) are satisfied.

Let $\{\varepsilon_i\}_{i=1}^\infty$ and $\{\delta_i\}_{i=1}^\infty$ be sequences with $\varepsilon_i, \delta_i \downarrow 0$ as $i \to \infty$. For Ω_0^i we define

$$r_{\varepsilon_i}(x) = \begin{cases} \operatorname{dist}(x, \Gamma_0^i), & x \in \Omega_0^i \\ -\operatorname{dist}(x, \Gamma_0^i), & x \notin \Omega_0^i. \end{cases}$$

We remark that $|\nabla r_{\varepsilon_i}| \leq 1$ a.e. $x \in \mathbb{R}^n$ and r_{ε_i} is smooth near Γ_0^i. Let $\overline{r_{\varepsilon_i}}$ be a smoothing of r_{ε_i} with $|\nabla\overline{r_{\varepsilon_i}}| \leq 1$, $|\nabla^2\overline{r_{\varepsilon_i}}| \leq \varepsilon_i^{-1}$ in $\mathbb{R}^n$ and $\overline{r_{\varepsilon_i}} = r_{\varepsilon_i}$ near Γ_0^i.

Define

$$\varphi_0^{\varepsilon_i} = q^{\varepsilon_i}(\overline{r_{\varepsilon_i}}(x)) \quad \text{and} \quad \varphi_0^{\varepsilon_i,\delta_j} = q^{\varepsilon_i,\delta_j}(\overline{r_{\varepsilon_i}}(x)), \quad i,j \geq 1. \tag{2.12}$$

Let $U \subset \mathbb{R}^n$ be a bounded open set and $Q_T := U \times (0,T)$ for $T > 0$. By (2.6), (2.8) and (2.12) there exists $c_1(i) > 0$ such that

$$\sup_{j\in\mathbb{N}} \|\varphi_0^{\varepsilon_i,\delta_j}\|_{C^2(\overline{U})} \leq c_1(i) \tag{2.13}$$

and

$$\lim_{j\to\infty} \|\varphi_0^{\varepsilon_i,\delta_j} - \varphi_0^{\varepsilon_i}\|_{C^{1,\alpha}(\overline{U})} = 0 \tag{2.14}$$

for $i \geq 1$. Let $\varphi^{\varepsilon_i,\delta_j}$ be a solution for (1.2) with initial data $\varphi_0^{\varepsilon_i,\delta_j}$. Then $\sup_{Q_T} |\varphi^{\varepsilon_i,\delta_j}| \leq \frac{1}{1-\delta_j}$ and $\sup_{Q_T} |F_{\delta_j}(\varphi^{\varepsilon_i,\delta_j})| \leq \max_{|s|\leq\frac{1}{1-\delta_j}} |F_{\delta_j}(s)| = 1$ by the maximal principle. Thus by (2.13) and the standard arguments for parabolic equations (see [21, p.517]), for any open set $U' \subset\subset U$ there exists $c_2(i) > 0$ such that

$$\sup_{j\in\mathbb{N}} \|\varphi^{\varepsilon_i,\delta_j}\|_{C^{1,\alpha}(\overline{Q_T'})} \leq c_2(i), \quad i \geq 1, \tag{2.15}$$

where $Q_T' := U' \times (0,T)$. Hence by (2.14), (2.15), the Arzelà-Ascoli theorem and the diagonal argument there exists a subsequence $\{\delta_j\}_{j=1}^\infty$ (denoted by the same index) such that for any compact set $K \subset \mathbb{R}^n$ and $T > 0$ we have

$$\varphi^{\varepsilon_i,\delta_j} \to \varphi^{\varepsilon_i} \quad \text{in} \quad C^{1,\alpha}(K \times [0,T]) \ \text{ and } \sup_{\mathbb{R}^n\times[0,T]} |\varphi^{\varepsilon_i}| \leq 1, \quad i \geq 1, \tag{2.16}$$

where φ^{ε_i} is a solution for (1.1) with initial data $\varphi_0^{\varepsilon_i}$ (see [7, Section 2]). Thus for $i \geq 1$ and any compact set $K \subset \mathbb{R}^n$ we have

$$e_{i,j} \to e_i \ \text{ uniformly on } K \times [0,T], \tag{2.17}$$

where $e_{i,j} = \frac{\varepsilon_i|\nabla\varphi^{\varepsilon_i,\delta_j}|^2}{2} + \frac{F_{\delta_j}(\varphi^{\varepsilon_i,\delta_j})}{\varepsilon_i}$, $e_i = \frac{\varepsilon_i|\nabla\varphi^{\varepsilon_i}|^2}{2} + \frac{F_0(\varphi^{\varepsilon_i})}{\varepsilon_i}$ and $F_0(s) = \frac{1-s^2}{2}$. Hence

$$\mu_t^{\varepsilon_i,\delta_j} \to \mu_t^{\varepsilon_i} \qquad \text{as Radon measures, } i \geq 1, \tag{2.18}$$

where $\mu_t^{\varepsilon_i}$ is a Radon measure defined by

$$\mu_t^{\varepsilon_i}(\phi) := \int_{\mathbb{R}^n} \phi\Big(\frac{\varepsilon_i|\nabla\varphi^{\varepsilon_i}|^2}{2} + \frac{F_0(\varphi^{\varepsilon_i})}{\varepsilon_i}\Big)dx \tag{2.19}$$

for any $\phi \in C_c(\mathbb{R}^n)$. By the definition of $\varphi_0^{\varepsilon_i,\delta_j}$ we obtain the following:

Proposition 2.6 (see Proposition 1.4 of [18]).

(1) There exists $D_1 = D_1(D_0) > 0$ such that for any $i, j \geq 1$, we have

$$\sup_{x\in\mathbb{R}^n, R>0} \Big\{\mu_0^{\varepsilon_i,\delta_j}(B_R(x)), \frac{\mu_0^{\varepsilon_i,\delta_j}(B_R(x))}{\omega_{n-1}R^{n-1}}\Big\} \leq D_1. \tag{2.20}$$

(2) $\lim_{i\to\infty}\mu_0^{\varepsilon_i} = \frac{\pi}{2}\mathcal{H}^{n-1}\lfloor\Gamma_0$ as Radon measures,

(3) $\lim_{i\to\infty}\varphi_0^{\varepsilon_i} = 2\chi_{\Omega_0^+} - 1$ in BV_{loc},

(4) for any $i, j \geq 1$ we have

$$\frac{\varepsilon_i|\nabla\varphi_0^{\varepsilon_i,\delta_j}|^2}{2} \leq \frac{F_{\delta_j}(\varphi_0^{\varepsilon_i,\delta_j})}{\varepsilon_i} \quad \text{on} \quad \mathbb{R}^n. \tag{2.21}$$

Proof. We only prove (2) and (4). In the same manner as [18] we have

$$\lim_{i\to\infty}\mu_0^{\varepsilon_i} = \lim_{\delta\downarrow 0}\sigma_\delta\mathcal{H}^{n-1}\lfloor\Gamma_0.$$

By $\lim_{\delta\downarrow 0}\sigma_\delta = \int_{-1}^1\sqrt{2F_0(s)}\,ds = \frac{\pi}{2}$ we obtain (2). We compute that

$$\frac{\varepsilon_i|\nabla\varphi_0^{\varepsilon_i,\delta_j}|^2/2}{F_{\delta_j}(\varphi_0^{\varepsilon_i,\delta_j})/\varepsilon_i} = \frac{\varepsilon_i(q_r^{\varepsilon_i,\delta_j})^2/2}{F_{\delta_j}(q^{\varepsilon_i,\delta_j})/\varepsilon_i}|\nabla\overline{r_{\varepsilon_i}}|^2 = |\nabla\overline{r_{\varepsilon_i}}|^2 \leq 1,$$

where (2.7) and $|\nabla\overline{r_{\varepsilon_i}}| \leq 1$ are used. Hence we obtain (2.21). □

Our main results are the following:

Theorem 2.7. Let $\Omega_0^+ \subset \mathbb{R}^n$ be a bounded open set and satisfy (2.10) and (2.11). Let $\varphi^{\varepsilon_i} \in C_{loc}^{1,\alpha}(\mathbb{R}^n\times(0,\infty))$ be a solution for (1.1) with initial data $\varphi_0^{\varepsilon_i}$, and $\varphi^{\varepsilon_i,\delta_j} \in C_{loc}^{2,\alpha}(\mathbb{R}^n\times(0,\infty))$ be a solution for (1.2) with initial data $\varphi_0^{\varepsilon_i,\delta_j}$, where $\varphi_0^{\varepsilon_i}$ and $\varphi_0^{\varepsilon_i,\delta_j}$ are defined by (2.12).

Then there exist

(a) subsequences $\{i_k\}_{k=1}^\infty$, $\{j_k\}_{k=1}^\infty$ and a family of Radon measures $\{\mu_t\}_{t\geq 0}$ such that

$$\mu_t^{\varepsilon_i,\delta_{j_k}} \to \mu_t^{\varepsilon_i} \quad \text{as} \quad k\to\infty, \quad t\geq 0,\ i\geq 1, \tag{2.22}$$

$$\mu_t^{\varepsilon_{i_k},\delta_{j_k}} \to \mu_t \quad \text{as} \quad k\to\infty, \quad t\geq 0, \tag{2.23}$$

$$\mu_t^{\varepsilon_{i_k}} \to \mu_t \quad \text{as} \quad k\to\infty, \quad t\geq 0 \tag{2.24}$$

and $\{\mu_t\}_{t\geq 0}$ is a global solution for Brakke's mean curvature flow with initial data $\mu_0 = \frac{\pi}{2}\mathcal{H}^{n-1}\lfloor\Gamma_0$,

(b) and $\varphi \in BV_{loc}(\mathbb{R}^n\times[0,\infty)) \cap C_{loc}^{\frac{1}{2}}([0,\infty); L^1(\mathbb{R}^n))$ such that

(b1) $\varphi^{\varepsilon_{i_k},\delta_{j_k}} \to 2\varphi - 1$ in $L_{loc}^1(\mathbb{R}^n\times[0,\infty))$ and a.e. pointwise,

(b2) $\varphi(\cdot,0) = \chi_{\Omega_0^+}$ a.e. on $\mathbb{R}^n$,

(b3) $\varphi(\cdot,t)$ is a characteristic function for all $t\in[0,\infty)$,

(b4) $\|\nabla\varphi(\cdot,t)\|(\phi) \leq \frac{2}{\pi}\mu_t(\phi)$ for any $t\in[0,\infty)$ and $\phi\in C_c(\mathbb{R}^n;\mathbb{R}^+)$. Moreover $\operatorname{spt}\|\nabla\varphi(\cdot,t)\| \subset \operatorname{spt}\mu_t$ for any $t\in[0,\infty)$.

3. Monotonicity formula

In this section, we consider the monotonicity formula for $\mu_t^{\varepsilon_i,\delta_j}$ and prove the negativity of the discrepancy measure which we define later. We assume that $\Omega_0^+ \subset \mathbb{R}^n$ is a bounded open set and satisfies (2.10) and (2.11), and $\varphi^{\varepsilon_i,\delta_j} \in C^{2,\alpha}_{loc}(\mathbb{R}^n \times (0,\infty))$ is a solution for (1.2) with initial data $\varphi_0^{\varepsilon_i,\delta_j}$, where $\varphi_0^{\varepsilon_i,\delta_j}$ is defined by (2.12) in this section. We denote ε_i and δ_j by ε and δ.

We define the backward heat kernel ρ by

$$\rho = \rho_{y,s}(x,t) := \frac{1}{(4\pi(s-t))^{\frac{n-1}{2}}} e^{-\frac{|x-y|^2}{4(s-t)}}, \qquad t<s,\ x,y\in\mathbb{R}^n.$$

We define a Radon measure $\xi_t^{\varepsilon,\delta}$ by

$$\xi_t^{\varepsilon,\delta}(\phi) := \int_{\mathbb{R}^n} \phi\Big(\frac{\varepsilon|\nabla\varphi^{\varepsilon,\delta}|^2}{2} - \frac{F_\delta(\varphi^{\varepsilon,\delta})}{\varepsilon}\Big)dx, \tag{3.1}$$

for any $\phi \in C_c(\mathbb{R}^n)$. $\xi_t^{\varepsilon,\delta}$ is called a discrepancy measure. The monotonicity formula for the mean curvature flow is proved by Huisken [17]. Ilmanen [18] proved the monotonicity formula for the Allen-Cahn equation without constraint. The following monotonicity formula is obtained in the same manner as [18, 3.3]. So we skip the proof.

Proposition 3.1.

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^n} \rho\, d\mu_t^{\varepsilon,\delta}(x) = &-\int_{\mathbb{R}^n} \varepsilon\rho\Big(-\Delta\varphi^{\varepsilon,\delta} + \frac{F_\delta'(\varphi^{\varepsilon,\delta})}{\varepsilon^2} - \frac{\nabla\varphi^{\varepsilon,\delta}\cdot\nabla\rho}{\rho}\Big)^2 d\mu_t^{\varepsilon,\delta}(x)\\ &+\frac{1}{2(s-t)}\int_{\mathbb{R}^n}\rho\, d\xi_t^{\varepsilon,\delta}(x)\end{aligned} \tag{3.2}$$

for $y\in\mathbb{R}^n$ and $0\le t<s$.

Define $\xi_{\varepsilon,\delta} = \xi_{\varepsilon,\delta}(x,t) := \frac{\varepsilon|\nabla\varphi^{\varepsilon,\delta}(x,t)|^2}{2} - \frac{F_\delta(\varphi^{\varepsilon,\delta}(x,t))}{\varepsilon}$.

Proposition 3.2. $\xi_{\varepsilon,\delta}(x,t)\le 0$ for any $(x,t)\in\mathbb{R}^n\times[0,\infty)$. Moreover $\xi_t^{\varepsilon,\delta}$ is a non-positive measure for $t\in[0,\infty)$.

Proof. Let $h>0$ and $F_{\delta,h}\in C^\infty(\mathbb{R})$ be a function with $\lim_{h\to 0}\|F_{\delta,h}-F_\delta\|_{C^1(\mathbb{R})}=0$.

Let $q^{\varepsilon,\delta,h}\in C^\infty(\mathbb{R})$ be a solution for

$$\frac{\varepsilon(q_r^{\varepsilon,\delta,h})^2}{2} = \frac{F_{\delta,h}(q^{\varepsilon,\delta,h})}{\varepsilon} \quad \text{on } \mathbb{R} \tag{3.3}$$

with $\lim_{h\to0}\|q^{\varepsilon,\delta}-q^{\varepsilon,\delta,h}\|_{C^2([-L,L])}=0$ for any $L>0$. We remark that we have

$$q_{rr}^{\varepsilon,\delta,h} = \frac{F_{\delta,h}'(q^{\varepsilon,\delta,h})}{\varepsilon^2} \quad \text{on } \mathbb{R}. \tag{3.4}$$

Let $\varphi^{\varepsilon,\delta,h}\in C^{2,\alpha}(\mathbb{R}\times(0,\infty))$ be a solution for

$$\begin{cases} \partial_t\varphi^{\varepsilon,\delta,h} - \Delta\varphi^{\varepsilon,\delta,h} + \dfrac{F_{\delta,h}'(\varphi^{\varepsilon,\delta,h})}{\varepsilon^2} = 0, & (x,t)\in\mathbb{R}^n\times(0,\infty),\\ \varphi^{\varepsilon,\delta,h}(x,0) = \varphi_0^{\varepsilon,\delta,h}(x), & x\in\mathbb{R}^n,\end{cases} \tag{3.5}$$

where $\varphi_0^{\varepsilon,\delta,h}$ is defined by

$$\varphi_0^{\varepsilon,\delta,h}(x) = q^{\varepsilon,\delta,h}(\overline{r_\varepsilon}(x)), \quad x\in\mathbb{R}^n.$$

We define a function $r : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ by

$$\varphi^{\varepsilon,\delta,h}(x,t) = q^{\varepsilon,\delta,h}(r(x,t)), \quad (x,t) \in \mathbb{R}^n \times [0,\infty).$$

By (3.3) we have

$$\frac{\varepsilon|\nabla\varphi^{\varepsilon,\delta,h}|^2/2}{F_{\delta,h}(\varphi^{\varepsilon,\delta,h})/\varepsilon} \leq |\nabla r|^2 \quad \text{on} \quad \mathbb{R}^n \times [0,\infty).$$

Hence, if $|\nabla r| \leq 1$ for any $h > 0$ then $\dfrac{\varepsilon|\nabla\varphi^{\varepsilon,\delta}|^2/2}{F_\delta(\varphi^{\varepsilon,\delta})/\varepsilon} \leq 1$. Thus we only need to prove that $|\nabla r| \leq 1$ on $\mathbb{R}^n \times [0,\infty)$.

Let $g^{\delta,h}(s) := \sqrt{2F_{\delta,h}(s)}$. By (3.3) and (3.4) we have

$$q_r^{\varepsilon,\delta,h} = \frac{g^{\delta,h}(q^{\varepsilon,\delta,h})}{\varepsilon} \qquad \text{and} \qquad q_{rr}^{\varepsilon,\delta,h} = \frac{(g^{\delta,h}(q^{\varepsilon,\delta,h}))_r}{\varepsilon} = \frac{g_q^{\delta,h}(q^{\varepsilon,\delta,h})}{\varepsilon} q_r^{\varepsilon,\delta,h}. \tag{3.6}$$

By (3.4), (3.5) and (3.6) we obtain

$$\begin{aligned} q_r^{\varepsilon,\delta,h}\partial_t r &= q_r^{\varepsilon,\delta,h}\Delta r + q_{rr}^{\varepsilon,\delta,h}|\nabla r|^2 - q_{rr}^{\varepsilon,\delta,h} \\ &= q_r^{\varepsilon,\delta,h}\Delta r + q_r^{\varepsilon,\delta,h}\frac{g_q^{\delta,h}}{\varepsilon}(|\nabla r|^2 - 1). \end{aligned} \tag{3.7}$$

Thus we have

$$\partial_t r = \Delta r + \frac{g_q^{\delta,h}}{\varepsilon}(|\nabla r|^2 - 1)$$

and

$$\partial_t|\nabla r|^2 = \frac{1}{2}\Delta|\nabla r|^2 - |\nabla^2 r|^2 + \frac{2}{\varepsilon}\nabla r \cdot \nabla g_q^{\delta,h}(|\nabla r|^2 - 1) + \frac{2}{\varepsilon}\nabla r \cdot \nabla|\nabla r|^2. \tag{3.8}$$

By the assumption we have $|\nabla r(\cdot,0)| = |\nabla \overline{r_\varepsilon}| \leq 1$ on $\mathbb{R}^n$. By (3.8) and the maximal principle we obtain $|\nabla r| \leq 1$ in $\mathbb{R}^n \times [0,\infty)$. □

By Proposition 3.1 and Proposition 3.2 we have

Proposition 3.3. For $y \in \mathbb{R}^n$ and $0 \leq t < s$ we have

$$\frac{d}{dt}\int_{\mathbb{R}^n} \rho\, d\mu_t^{\varepsilon,\delta}(x) \leq -\int_{\mathbb{R}^n} \varepsilon\rho\Big(-\Delta\varphi^{\varepsilon,\delta} + \frac{F_\delta'(\varphi^{\varepsilon,\delta})}{\varepsilon^2} - \frac{\nabla\varphi^{\varepsilon,\delta}\cdot\nabla\rho}{\rho}\Big)^2 d\mu_t^{\varepsilon,\delta} \leq 0. \tag{3.9}$$

Next we prove the upper density ratio bounds of $\mu_t^{\varepsilon,\delta}$.

Proposition 3.4. There exists $c_3 = c_3(n) > 0$ such that

$$\mu_t^{\varepsilon,\delta}(B_R(x)) \leq c_3 D_1 R^{n-1} \tag{3.10}$$

for $(x,t) \in \mathbb{R}^n \times [0,\infty)$ and $R > 0$.

Proof. We compute that

$$\begin{aligned} &\int_{\mathbb{R}^n} \rho_{y,s}(x,0)\, d\mu_0^{\varepsilon,\delta}(x) = \frac{1}{(4\pi s)^{\frac{n-1}{2}}}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4s}}\, d\mu_0^{\varepsilon,\delta} \\ =&\frac{1}{(4\pi s)^{\frac{n-1}{2}}}\int_0^1 \mu_0^{\varepsilon,\delta}(\{x \,|\, e^{-\frac{|x-y|^2}{4s}} > k\})\, dk = \frac{1}{(4\pi s)^{\frac{n-1}{2}}}\int_0^1 \mu_0^{\varepsilon,\delta}(B_{\sqrt{4s\log k^{-1}}}(y))\, dk \\ \leq&\frac{1}{(4\pi s)^{\frac{n-1}{2}}}\int_0^1 D_1\omega_{n-1}(\sqrt{4s\log k^{-1}})^{n-1}\, dk \leq c_4 D_1, \end{aligned} \tag{3.11}$$

where $c_4 > 0$ is depending only on n and the density upper bound (2.20) is used. By the monotonicity formula (3.9), we have

$$\int_{\mathbb{R}^n} \rho_{y,s}(x,t)\, d\mu_t^{\varepsilon,\delta}(x) \leq \int_{\mathbb{R}^n} \rho_{y,s}(x,0)\, d\mu_0^{\varepsilon,\delta}(x) \leq c_4 D_1, \tag{3.12}$$

for any $0 < t < s$ and $y \in \mathbb{R}^n$. Fix $R > 0$ and set $s = t + \frac{R^2}{4}$. Then

$$\begin{aligned} \int_{\mathbb{R}^n} \rho_{y,s}(x,t)\, d\mu_t^{\varepsilon,\delta} &= \int_{\mathbb{R}^n} \frac{1}{\pi^{\frac{n-1}{2}} R^{n-1}} e^{-\frac{|x-y|^2}{R^2}}\, d\mu_t^{\varepsilon,\delta} \geq \int_{B_R(y)} \frac{1}{\pi^{\frac{n-1}{2}} R^{n-1}} e^{-\frac{|x-y|^2}{R^2}}\, d\mu_t^{\varepsilon,\delta} \\ \geq \int_{B_R(y)} \frac{1}{\pi^{\frac{n-1}{2}} R^{n-1}} e^{-1}\, d\mu_t^{\varepsilon,\delta} &= \frac{1}{e\pi^{\frac{n-1}{2}} R^{n-1}} \mu_t^{\varepsilon,\delta}(B_R(y)). \end{aligned} \tag{3.13}$$

By (3.12) and (3.13) we obtain (3.10). $\square$

4. Existence of limit measures

In this section, we prove the existence of limit measure μ_t. We also assume that $\Omega_0^+ \subset \mathbb{R}^n$ satisfies (2.10) and (2.11), $\varphi^{\varepsilon_i,\delta_j} \in C^{2,\alpha}_{loc}(\mathbb{R}^n \times (0,\infty))$ is a solution for (1.2) with initial data $\varphi_0^{\varepsilon_i,\delta_j}$, where $\varphi_0^{\varepsilon_i,\delta_j}$ is defined by (2.12), and (2.16) and (2.18) hold in this section.

Lemma 4.1. For any $\phi \in C_c^2(\mathbb{R}^n;\mathbb{R}^+)$, $i,j \geq 1$ and $t > 0$ we have

$$\frac{d}{dt}\mu_t^{\varepsilon_i,\delta_j}(\phi) \leq \sup_{x\in\mathbb{R}^n} |\nabla^2\phi| \mu_t^{\varepsilon_i,\delta_j}(\operatorname{spt}\phi). \tag{4.1}$$

Moreover there exists $c_5 = c_5(n, D_1, \operatorname{spt}\phi, \sup_{x\in\mathbb{R}^n}|\nabla^2\phi|) > 0$ such that the function $\mu_t^{\varepsilon_i}(\phi) - c_5 t$ of t is nonincreasing for any $i \geq 1$.

Proof. We denote ε_i, δ_j and $\varphi^{\varepsilon_i,\delta_j}$ by ε, δ and φ. By the integration by parts,

$$\begin{aligned} &\frac{d}{dt}\mu_t^{\varepsilon,\delta}(\phi) = \int_{\mathbb{R}^n} \phi \frac{\partial}{\partial t}\Big(\frac{\varepsilon|\nabla\varphi|^2}{2} + \frac{F_\delta(\varphi)}{\varepsilon}\Big)\, dx \\ =&\int_{\mathbb{R}^n} \phi\Big(\varepsilon\nabla\varphi\cdot\nabla\varphi_t + \frac{F_\delta'(\varphi)}{\varepsilon}\varphi_t\Big)\, dx \\ =&\int_{\mathbb{R}^n} \varepsilon\phi\Big(-\Delta\varphi + \frac{F_\delta'(\varphi)}{\varepsilon^2}\Big)\varphi_t - \varepsilon(\nabla\phi\cdot\nabla\varphi)\varphi_t\, dx \\ =&\int_{\mathbb{R}^n} -\varepsilon\phi\Big(-\Delta\varphi + \frac{F_\delta'(\varphi)}{\varepsilon^2}\Big)^2 + \varepsilon(\nabla\phi\cdot\nabla\varphi)\Big(-\Delta\varphi + \frac{F_\delta'(\varphi)}{\varepsilon^2}\Big)\, dx \\ =&\int_{\mathbb{R}^n} -\varepsilon\phi\Big(-\Delta\varphi + \frac{F_\delta'(\varphi)}{\varepsilon^2} - \frac{\nabla\phi\cdot\nabla\varphi}{2\phi}\Big)^2 + \varepsilon\frac{(\nabla\phi\cdot\nabla\varphi)^2}{4\phi}\, dx \\ \leq&\Big(\sup_{x\in\{x\,|\,\phi(x)>0\}} \frac{|\nabla\phi|^2}{2\phi}\Big)\mu^{\varepsilon,\delta}(\operatorname{spt}\phi) \leq \sup_{x\in\mathbb{R}^n}|\nabla^2\phi|\mu^{\varepsilon,\delta}(\operatorname{spt}\phi), \end{aligned} \tag{4.2}$$

where $\sup_{x\in\{x\,|\,\phi(x)>0\}}\frac{|\nabla\phi|^2}{2\phi} \leq \sup_{x\in\mathbb{R}^n}|\nabla^2\phi|$ are used. By (3.10) and (4.2) there exists $c_5 = c_5(n, D_1, \operatorname{spt}\phi, \sup_{x\in\mathbb{R}^n}|\nabla^2\phi|) > 0$ such that $\mu_t^{\varepsilon,\delta}(\phi) - c_5 t$ of t is nonincreasing. By $\mu_t^{\varepsilon,\delta} \to \mu_t^\varepsilon$ for any $\varepsilon > 0$, $\mu_t^\varepsilon(\phi) - c_5 t$ of t is also nonincreasing. $\square$

Remark 4.2. By an argument similar to that in the proof of Lemma 4.1, for any $i,j \geq 1$ and $T \geq 0$ we have

$$\mu_T^{\varepsilon_i,\delta_j}(\mathbb{R}^n) + \int_0^T \int_{\mathbb{R}^n} \varepsilon_i\Big(-\Delta\varphi^{\varepsilon_i,\delta_j} + \frac{F_{\delta_j}'(\varphi^{\varepsilon_i,\delta_j})}{\varepsilon_i^2}\Big)^2\, dxdt = \mu_0^{\varepsilon_i,\delta_j}(\mathbb{R}^n) \leq D_1. \tag{4.3}$$

Proposition 4.3. There exist subsequences $\{\varepsilon_{i_k}\}_{k=1}^\infty$, $\{\delta_{j_k}\}_{k=1}^\infty$ and a family of Radon measures $\{\mu_t\}_{t\geq 0}$ such that (2.22), (2.23) and (2.24) hold.

Proof. By (2.18) we only need to prove (2.23) and (2.24). First we prove that there exist $\{\varepsilon_{i_k}\}_{k=1}^\infty$ and a family of Radon measures $\{\mu_t\}_{t\geq 0}$ such that $\mu_t^{\varepsilon_{i_k}} \to \mu_t$ as $k\to\infty$ for any $t\geq 0$.

Let $B_1 \subset [0,\infty)$ be a countable dense set. By the compactness of Radon measures, there exists a subsequence $\{\varepsilon_{i_k}\}_{k=1}^\infty$ and a family of Radon measures $\{\mu_t\}_{t\in B_1}$ such that $\mu_t^{\varepsilon_{i_k}} \to \mu_t$ as $k\to\infty$ for any $t\in B_1$. Let $\{\phi_l\}_{l=1}^\infty \subset C_c^2(\mathbb{R}^n;\mathbb{R}^+)$ be a dense set.

By Lemma 4.1, $\mu_t(\phi_l) - c_5(\phi_l)t$ of $t\in B_1$ is nonincreasing for any $l\geq 1$. Hence for any $l\geq 1$ there exists a countable set $E_l\subset[0,\infty)$ such that

$$\lim_{t\uparrow s, t\in B_1} \mu_t(\phi_l) = \lim_{t\downarrow s, t\in B_1} \mu_t(\phi_l) \tag{4.4}$$

for any $s\in[0,\infty)\setminus E_l$. Set $B_2 = [0,\infty)\setminus \cup_l E_l$. Then B_2 is co-countable and (4.4) holds for any $l\geq 1$ and $s\in B_2$.

Let $s\in B_2\setminus B_1$. By the compactness of Radon measures, there exist a subsequence $\{\varepsilon_{i_{k_m}}\}_{m=1}^\infty$ and a Radon measure μ_s such that $\mu_s^{\varepsilon_{i_{k_m}}} \to \mu_s$ as $m\to\infty$.

Next we show that μ_s is unique and $\mu_s^{\varepsilon_{i_k}} \to \mu_s$ as $k\to\infty$. By Lemma 4.1, for any $l\geq 1$, $m\in\mathbb{N}$ and t_1,t_2 with $t_1<s<t_2$ we have

$$\mu_{t_1}^{\varepsilon_{i_{k_m}}}(\phi_l) - c_5(t_1-s) \geq \mu_s^{\varepsilon_{i_{k_m}}}(\phi_l) \geq \mu_{t_2}^{\varepsilon_{i_{k_m}}}(\phi_l) - c_5(t_2-s).$$

Hence for any $t_1,t_2\in B_1$ with $t_1<s<t_2$ we have

$$\mu_{t_1}(\phi_l) - c_5(t_1-s) \geq \mu_s(\phi_l) \geq \mu_{t_2}(\phi_l) - c_5(t_2-s).$$

Therefore by (4.4) we obtain $\mu_s(\phi_l) = \lim_{t\uparrow s, t\in B_1}\mu_t(\phi_l) = \lim_{t\downarrow s, t\in B_1}\mu_t(\phi_l)$ for any $l\geq 1$. Thus μ_s is uniquely determined. Moreover, $\mu_s^{\varepsilon_{i_k}} \to \mu_s$ as $k\to\infty$.

Therefore $\mu_t^{\varepsilon_{i_k}} \to \mu_t$ as $k\to\infty$ for any $t\in B_1\cup B_2$. Because $[0,\infty)\setminus(B_1\cup B_2)$ is a countable set, there exists a subsequence $\{\varepsilon_{i_k}\}_{k=1}^\infty$ (denoted by the same index) such that $\mu_t^{\varepsilon_{i_k}} \to \mu_t$ as $k\to\infty$ for any $t\in[0,\infty)$.

Next we show that there exists a subsequences $\{\delta_{j_k}\}_{k=1}^\infty$ such that $\mu_t^{\varepsilon_{i_k},\delta_{j_k}} \to \mu_t$ as $k\to\infty$ for any $t\geq 0$. For $\phi\in C_c(\mathbb{R}^n)$ we compute that

$$\begin{aligned} |\mu_t^{\varepsilon_{i_k},\delta_j}(\phi) - \mu_t(\phi)| &\leq |\mu_t^{\varepsilon_{i_k},\delta_j}(\phi) - \mu_t^{\varepsilon_{i_k}}(\phi)| + |\mu_t^{\varepsilon_{i_k}}(\phi) - \mu_t(\phi)| \\ &\leq \sup_{\mathbb{R}^n}|\phi| \int_{\operatorname{spt}\phi} |e_{i_k,j}(x,t) - e_{i_k}(x,t)|\,dx + |\mu_t^{\varepsilon_{i_k}}(\phi) - \mu_t(\phi)|. \end{aligned} \tag{4.5}$$

Let $\{R_k\}_{k=1}^\infty$ and $\{T_k\}_{k=1}^\infty$ satisfy $R_k, T_k\to\infty$ as $k\to\infty$. By (2.17) and the diagonal argument, there exists a subsequence $\{\delta_{j_k}\}_{k=1}^\infty$ such that

$$\sup_{t\in[0,T_k]} \int_{B_{R_k}(0)} |e_{i_k,j_k}(x,t) - e_{i_k}(x,t)|\,dx \leq \frac{1}{k} \quad \text{for} \quad k\geq 1. \tag{4.6}$$

By (4.5) and (4.6) we have $\mu_t^{\varepsilon_{i_k},\delta_{j_k}} \to \mu_t$ as $k\to\infty$ for any $t\geq 0$. Hence we obtain (2.22), (2.23) and (2.24). ☐

5. Forward density lower bound and vanishing of ξ

In this section we prove the lower density estimate for μ_t and the vanishing of ξ by using the technique of Ilmanen [18] and Takasao and Tonegawa [25]. Assume that $\varphi^{\varepsilon_i,\delta_j}$ and $\mu_t^{\varepsilon_i,\delta_j}$

satisfy all the assumptions of Section 2 and $\mu_t^i := \mu_t^{\varepsilon_i,\delta_i} \to \mu_t$ and $\xi_t^i := \xi_t^{\varepsilon_i,\delta_i} \to \xi_t$ for any $t \geq 0$ as Radon measures in this section. We denote $\varphi^i := \varphi^{\varepsilon_i,\delta_i}$ in this section.

By the computation we have the following estimates. The proof is omitted.

Lemma 5.1. We have

$$F_\delta'(s)\Big(s - \frac{1}{1-\delta}\Big) \geq \frac{1}{100} F_\delta(s) \tag{5.1}$$

for any $\delta \in (0, \frac{3}{10})$ and $s \in [\frac{3}{4}, \infty)$, and

$$F_\delta'(s)\Big(s + \frac{1}{1-\delta}\Big) \geq \frac{1}{100} F_\delta(s) \tag{5.2}$$

for any $\delta \in (0, \frac{3}{10})$ and $s \in (-\infty, -\frac{3}{4}]$.

Let μ be a measure on $\mathbb{R}^n \times [0,\infty)$ such that $d\mu = d\mu_t dt$.

Lemma 5.2. Assume $(x',t') \in \operatorname{spt}\mu$. Then there exist a subsequence $\{i_j\}_{j=1}^\infty$ and $\{(x_j,t_j)\}_{j=1}^\infty$ such that

$$\lim_{j\to\infty}(x_j,t_j) = (x',t') \qquad \text{and} \qquad |\varphi^{i_j}(x_j,t_j)| < \frac{3}{4} \tag{5.3}$$

for any $j \in \mathbb{N}$.

Proof. Set $Q_r = B_r(x') \times [t'-r^2, t'+r^2]$ for $r > 0$. If the claim were not true, then there exist $r > 0$ and $N \in \mathbb{N}$ such that $\inf_{Q_r} |\varphi^i| \geq \frac{3}{4}$ for any $i \geq N$. So we may assume that $\inf_{Q_r} \varphi^i \geq \frac{3}{4}$ for any $i \geq N$ without loss of generality. Moreover we may assume $\delta_i \in (0, \frac{3}{10})$ for $i \geq N$. Let $\phi \in C_c^2(Q_r)$. Then by Lemma 5.1 and $\sup_{Q_r} |\varphi^i| \leq \frac{1}{1-\delta^i} \leq 2$ we have

$$\begin{aligned}
&\frac{1}{100}\int_{Q_r} \phi^2 \frac{F_{\delta_i}(\varphi^i)}{\varepsilon_i^2}\,dxdt \leq \int_{Q_r} \phi^2 \frac{F_{\delta_i}'(\varphi^i)}{\varepsilon_i^2}\Big(\varphi^i - \frac{1}{1-\delta_i}\Big)\,dxdt \\
\leq& \int_{Q_r} \phi^2(-\varphi_t^i + \Delta\varphi^i)(\varphi^i - \frac{1}{1-\delta_i})\,dxdt \\
=& \int_{t'-r^2}^{t+r^2} \frac{d}{dt}\Big(\int_{B_r(x')} \phi^2\Big(-\frac{1}{2}(\varphi^i)^2 + \frac{1}{1-\delta_i}\varphi^i\Big)\,dx\Big)dt \\
&+ \int_{Q_r} \frac{2}{1-\delta_i}\phi\nabla\phi\cdot\nabla\varphi^i - \phi^2|\nabla\varphi^i|^2 - 2\phi\varphi^i\nabla\phi\cdot\nabla\varphi^i\,dxdt \\
\leq& C(\phi) + \int_{Q_r} -\phi^2|\nabla\varphi^i|^2 + \frac{1}{2}\phi^2|\nabla\varphi^i|^2 + 4|\nabla\phi|^2\,dxdt \leq C(\phi),
\end{aligned} \tag{5.4}$$

where $C(\phi) > 0$ depends only on $\sup_{x\in\mathbb{R}^n}\{|\phi|, |\nabla\phi|\}$. By Proposition 3.2 and (5.4) we obtain

$$\int_{t'-r^2}^{t'+r^2}\int_{B_r(x')} \phi^2\,d\mu_t^i dt \leq 2\int_{Q_r} \phi^2 \frac{F_{\delta_i}(\varphi^i)}{\varepsilon_i}\,dxdt \leq 200C(\phi)\varepsilon_i.$$

Hence we have

$$\int_{Q_r} \phi^2\,d\mu = 0.$$

This proves that $(x',t') \notin \operatorname{spt}\mu$. $\square$

Set

$$\rho_y^r(x) := \frac{1}{(\sqrt{2\pi}r)^{n-1}} e^{-\frac{|x-y|^2}{2r^2}}, \qquad r > 0,\ x,y \in \mathbb{R}^n.$$

Note that $\rho_{y,s}(x,t) = \rho_y^r(x)$ for $r = \sqrt{2(s-t)}$. We use the following estimates.

Lemma 5.3 (See[18])**.** Let $D>0$ and ν be a measure satisfying $\sup_{R>0,x\in\mathbb{R}^n}\frac{\nu(B_R(x))}{\omega_{n-1}R^{n-1}}\le D$. Then the following hold:

(1) For any $a>0$ there is $\gamma_1=\gamma_1(a)>0$ such that for any $r>0$ and $x,x_1\in\mathbb{R}^n$ with $|x-x_1|\le\gamma_1 r$ we have
$$\int_{\mathbb{R}^n}\rho^r_{x_1}(y)\,d\nu(y)\le(1+a)\int_{\mathbb{R}^n}\rho^r_x\,d\nu(y)+aD. \tag{5.5}$$

(2) For any $r,R>0$ and $x\in\mathbb{R}^n$ we have
$$\int_{\mathbb{R}^n\setminus B_R(x)}\rho^r_x(y)\,d\nu(y)\le 2^{n-1}e^{-3R^2/8r^2}D. \tag{5.6}$$

(3) For any $a>0$ there is $\gamma_2=\gamma_2(a)>0$ such that for any $r,R>0$ with $1\le\frac{R}{r}\le1+\gamma_2$ and any $x\in\mathbb{R}^n$, we have
$$\int_{\mathbb{R}^n}\rho^R_x(y)\,d\nu(y)\le(1+a)\int_{\mathbb{R}^n}\rho^r_x(y)\,d\nu(y)+aD. \tag{5.7}$$

Lemma 5.4. There exist $\eta=\eta(n)>0$ and $\gamma_3=\gamma_3(n,D_1)>0$ with the following property. Given $0\le t<s$, define $r=\sqrt{2(s-t)}$ and $t'=s+\frac{r^2}{2}$. If $x\in\mathbb{R}^n$ satisfies
$$\int_{\mathbb{R}^n}\rho_{y,s}(x,t)\,d\mu_s(y)<\eta, \tag{5.8}$$
then $(\bar B_{\gamma_3 r}(x)\times\{t'\})\cap\operatorname{spt}\mu=\emptyset$.

Proof. Assume for a contradiction that $(x',t')\in\operatorname{spt}\mu$ for some $x'\in\bar B_{\gamma_3 r}(x)$ with (5.8), where γ_3 will be chosen later. Then by Lemma 5.2 there exist a sequence $\{(x_j,t_j)\}_{j=1}^\infty$ and $\{i_j\}_{j=1}^\infty$ such that $\lim_{j\to\infty}(x_j,t_j)=(x',t')$ and $|\varphi^{i_j}(x_j,t_j)|<\frac34$ for any j. By Proposition 3.2 and $\sup_{\mathbb{R}^n\times[0,\infty)}|\varphi^i|\le\frac{1}{1-\delta_i}$ for any $i\ge1$, we have
$$\sup_{\mathbb{R}^n\times[0,\infty)}|\nabla\varphi^i|\le\sup_{\mathbb{R}^n\times[0,\infty)}\frac{\sqrt{2F_{\delta_i}(\varphi^i)}}{\varepsilon_i}\le\frac{1}{\varepsilon_i}\quad\text{for}\quad i\ge1. \tag{5.9}$$
Thus, there exists $N\ge1$ such that
$$|\varphi^{i_j}(y,t_j)|\le\frac78\qquad\text{and}\qquad F_{\delta_{i_j}}(\varphi^{i_j}(y,t_j))\ge\frac{1}{10} \tag{5.10}$$
for any $y\in\bar B_{\varepsilon_{i_j}/8}(x_j)$ and $j>N$. Hence there exists $\eta=\eta(n)>0$ such that
$$2\eta\le\int_{\bar B_{\varepsilon_{i_j}/8}(x_j)}\frac{F_{\delta_{i_j}}(\varphi^{i_j}(y,t_j))}{\varepsilon_{i_j}}\rho_{x_j,t_j+\varepsilon_{i_j}^2}(y,t_j)\,dy\le\int_{\mathbb{R}^n}\rho_{x_j,t_j+\varepsilon_{i_j}^2}(y,t_j)\,d\mu^{i_j}_{t_j}(y), \tag{5.11}$$
where $\inf_{y\in\bar B_{\varepsilon_{i_j}/8}(x_j)}\rho_{x_j,t_j+\varepsilon_{i_j}^2}(y,t_j)\ge\frac{1}{(4\pi)^{\frac{n-1}{2}}\varepsilon_{i_j}^{n-1}e^{\frac{1}{256}}}>0$ is used. By the monotonicity formula (3.9) we have
$$\int_{\mathbb{R}^n}\rho_{x_j,t_j+\varepsilon_{i_j}^2}(y,t_j)\,d\mu^{i_j}_{t_j}(y)\le\int_{\mathbb{R}^n}\rho_{x_j,t_j+\varepsilon_{i_j}^2}(y,s)\,d\mu^{i_j}_s(y) \tag{5.12}$$
for sufficiently large j. Hence we obtain
$$2\eta\le\int_{\mathbb{R}^n}\rho_{x',t'}(y,s)\,d\mu_s(y). \tag{5.13}$$

By (2.20) and Lemma 5.3, for any $a > 0$ there exists $\gamma_1 = \gamma_1(a) > 0$ such that for any $x \in \bar{B}_{\gamma_1 r}(x')$ we have

$$\begin{aligned} 2\eta &\le \int_{\mathbb{R}^n} \rho_{x',t'}(y,s)\, d\mu_s(y) \le (1+a)\int_{\mathbb{R}^n} \rho_{x,t'}(y,s)\, d\mu_s(y) + ac_3D_1 \\ &= (1+a)\int_{\mathbb{R}^n} \rho_{x,s}(y,t)\, d\mu_s(y) + ac_3D_1 \le (1+a)\eta + ac_3D_1. \end{aligned} \tag{5.14}$$

We remark that $\rho_{x,t'}(y,s) = \rho_{x,s}(y,t)$ by $t'-s = s-t = \frac{r^2}{2}$. Set $a := \min\{\frac{1}{4}, \frac{\eta}{4c_3D_1}\}$ and $\gamma_3 := \gamma_1(a)$. Note that γ_3 depends only on n and D_1. Then we have $\eta < 0$. This is a contradiction to (5.8). Hence $(x',t') \notin \operatorname{spt}\mu$. ☐

Lemma 5.5. Let $U \subset \mathbb{R}^n$ be open. There exists $c_6 = c_6(n, D_1) > 0$ such that

$$\mathcal{H}^{n-1}(\operatorname{spt}\mu_t \cap U) \le c_6 \liminf_{r\to 0} \mu_{t-r^2}(U) \quad \text{for} \quad t > 0. \tag{5.15}$$

Proof. We only need to prove (5.15) for every compact set $K \subset U$. Let $X_t := \operatorname{spt}\mu_t \cap K$. By an argument similar to that in Lemma 5.4, for any $(x,t) \in X_t$ we have

$$2\eta \le \int_{\mathbb{R}^n} \rho_{x,t}(y, t-r^2)\, d\mu_{t-r^2}(y) \tag{5.16}$$

for sufficiently small $r > 0$. By (5.6), for any $L > 0$ we obtain

$$\int_{\mathbb{R}^n\setminus B_{rL}(x)} \rho_{x,t}(y,t-r^2)\, d\mu_{t-r^2}(y) \le 2^{n-1} e^{-\frac{3L^2}{8}} c_3 D_1.$$

Hence there exists $L = L(n, D_1) > 0$ such that

$$\eta \le \int_{B_{rL}(x)} \rho_{x,t}(y,t-r^2)\, d\mu_{t-r^2}(y).$$

Thus, by $\rho_{x,t}(\cdot, t-r^2) \le \frac{1}{(4\pi)^{\frac{n-1}{2}} r^{n-1}}$ we obtain

$$(4\pi)^{\frac{n-1}{2}} r^{n-1}\eta \le \mu_{t-r^2}(B_{rL}(x)). \tag{5.17}$$

Set $\mathcal{B} = \{\bar{B}_{rL}(x) \subset U \,|\, x \in X_t\}$. Note that $\mathcal{B}$ is a covering of X_t by closed balls centered at $x \in X_t$. By the Besicovitch covering theorem, there exists a finite sub-collection $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_{B(n)}$ such that each $\mathcal{B}_i$ is a disjoint set of closed balls and

$$X_t \subset \cup_{i=1}^{B(n)} \cup_{\bar{B}_{rL}(x_j)\in\mathcal{B}_i} \bar{B}_{rL}(x_j). \tag{5.18}$$

By (5.17) and (5.18) we obtain

$$\begin{aligned} \mathcal{H}^{n-1}_{rL}(X_t) &\le \sum_{i=1}^{B(n)} \sum_{\bar{B}_{rL}(x_j)\in\mathcal{B}_i} \omega_{n-1}(rL)^{n-1} \le \frac{\omega_{n-1}L^{n-1}}{(4\pi)^{\frac{n-1}{2}}\eta} \sum_{i=1}^{B(n)} \sum_{\bar{B}_{rL}(x_j)\in\mathcal{B}_i} \mu_{t-r^2}(\bar{B}_{rL}(x_j)) \\ &\le \frac{\omega_{n-1}L^{n-1}}{(4\pi)^{\frac{n-1}{2}}\eta} \sum_{i=1}^{B(n)} \mu_{t-r^2}(U) \le \frac{\omega_{n-1}L^{n-1}}{(4\pi)^{\frac{n-1}{2}}\eta} B(n)\mu_{t-r^2}(U), \end{aligned}$$

where $\mathcal{H}^{n-2+a}_{rL}$ is the approximate Hausdorff measure of $\mathcal{H}^{n-2+a}$. Set $c_6 := \frac{\omega_{n-1}L^{n-1}}{(4\pi)^{\frac{n-1}{2}}\eta} B(n)$ which depends only on n and D_1. Hence we obtain (5.15). ☐

Lemma 5.6. Let η be as in Lemma 5.4. Define

$$Z := \Big\{ (x,t) \in \operatorname{spt}\mu \,\Big|\, t \geq 0,\ \limsup_{s\downarrow t} \int \rho_{y,s}(x,t)\, d\mu_s(y) < \eta/2 \Big\}$$

and

$$Z_t := Z \cap (\mathbb{R}^n \times \{t\}) \quad \text{for} \quad t \geq 0.$$

Then for $a>0$, $\mathcal{H}^{n-2+a}(Z_t) = 0$ for a.e. $t \geq 0$. Moreover we have $\mu(Z) = 0$.

Proof. Let $a > 0$ and

$$Z^\tau := \left\{ (x,t) \in \operatorname{spt}\mu \,\middle|\, t \geq 0,\ \int \rho_{y,s}(x,t)\, d\mu_s(y) < \eta \text{ for all } s \in (t, t+\tau] \right\}.$$

First we prove $\mathcal{H}^{n-2+a}(Z_t) = 0$ for a.e. $t \geq 0$. Note that $Z \subset \cup_{m=1}^\infty Z^{\tau_m}$ for some $\{\tau_m\}_{m=1}^\infty$ with $\tau_m \in (0,1)$ and $\lim_{m\to\infty} \tau_m = 0$. So we only need to prove $\mathcal{H}^{n-2+a}(Z_t^\tau) = 0$ for any $\tau \in (0,1)$, where $Z^\tau_t := Z^\tau \cap (\mathbb{R}^n \times \{t\})$. Set $r := \sqrt{2(s-t)}$ and $t' := s + \frac{r^2}{2}$. For $(x,t) \in Z^\tau$, let $(x',t') \in \mathbb{R}^n \times [0,\infty)$ satisfy $|t'-t| \leq 2\tau$ and $|x'-x| \leq \gamma_3 r$. Then $(x',t') \notin \operatorname{spt}\mu \subset Z^\tau$ by Lemma 5.4 and $s - t \leq \tau$. Moreover, if $(x',t') \in Z^\tau$ then $\int \rho_{y,s}(x,t)\, d\mu_s(y) \geq \eta$ for any $x \in \bar{B}_{\gamma_3 r}(x')$ by $(x',t') \in \operatorname{spt}\mu$ and Lemma 5.4. Therefore the relation

$$|t'-t| \leq 2\tau \quad \text{and} \quad |x'-x| \leq \gamma_3 r$$

implies either $(x,t) \notin Z^\tau$ or $(x',t') \notin Z^\tau$. Hence for $(x,t) \in Z^\tau$ we have

$$P_{2\tau}(x,t) \cap Z^\tau = \{(x,t)\}. \tag{5.19}$$

Here, $P_{2\tau}(x,t)$ is defined by

$$P_{2\tau}(x,t) := \Big\{ (x',t') \,\Big|\, 2\tau \geq |t'-t| \geq \frac{|x'-x|^2}{\gamma_3^2} \Big\}.$$

Set

$$Z^{\tau,x_0,t_0} := Z^\tau \cap (B_1(x_0) \times [t_0-\tau, t_0+\tau]), \qquad x_0 \in \mathbb{R}^n,\ t_0 \geq 0.$$

Then there exists a countable set $K \subset \mathbb{R}^n \times [0,\infty)$ such that $Z^\tau \subset \cup_{(x_0,t_0)\in K} Z^{\tau,x_0,t_0}$. Hence we only need to prove $\mathcal{H}^{n-2+a}(Z_t^{\tau,x_0,t_0}) = 0$ for a.e. $t \in (0,\infty)$, where $Z_t^{\tau,x_0,t_0} = Z^{\tau,x_0,t_0} \cap (\mathbb{R}^n \times \{t\})$. Remark that for any $x \in \mathbb{R}^n$ the set $\{x\} \times [t_0-\tau, t_0+\tau] \cap Z^{\tau,x_0,t_0}$ has no more than one elements by (5.19). Define $P : \mathbb{R}^{n+1} \to \mathbb{R}^n$ by $P(x,t) = (x,0)$, where $(x,t) \in \mathbb{R}^n \times [0,\infty)$. Let $\delta' > 0$ and cover the projection $P(Z^{\tau,x_0,t_0}) \subset B_1(x_0) \times \{0\}$ by $\{B_{r_i}(x_i)\}_{i=1}^\infty$, where $(x_i,0) \in P(Z^{\tau,x_0,t_0})$, $r_i \leq \delta'$ and

$$\sum_{i=1}^\infty \omega_n r_i^n \leq 2\mathcal{L}^n(B_1(x_0)).$$

Let (x_i,t_i) be the point in Z^{τ,x_0,t_0} corresponding to x_i. By (5.19) we have $Z^{\tau,x_0,t_0} \subset \sum_{t_i\in[t_0-\tau,t_0+\tau]} B_{r_i}(x_i) \times [t_i - \frac{r_i^2}{\gamma_3^2}, t_i + \frac{r_i^2}{\gamma_3^2}]$. We compute that

$$\int_{t_0-\tau}^{t_0+\tau} \mathcal{H}^{n-2+a}_{\delta'}(Z_t^{\tau,x_0,t_0})\, dt \leq \int_{t_0-\tau}^{t_0+\tau} \sum_{i=1}^\infty \sum_{t\in[t_i-r_i^2/\gamma_3^2, t_i+r_i^2/\gamma_3^2]} \omega_{n-2+a} r_i^{n-2+a}\, dt$$

$$= \sum_{i=1}^\infty \int_{t_i-r_i^2/\gamma_3^2}^{t_i+r_i^2/\gamma_3^2} \omega_{n-2+a} r_i^{n-2+a}\, dt = \sum_{i=1}^\infty \frac{2\omega_{n-2+a}}{\gamma_3^2} r_i^{n+a} \leq \frac{4\omega_{n-2+a}}{\gamma_3^2 \omega_n} (\delta')^a \mathcal{L}^n(B_1(x_0)),$$

where $\mathcal{H}^{n-2+a}_{\delta'}$ is the approximate Hausdorff measure of $\mathcal{H}^{n-2+a}$. Then $\delta'\to 0$ implies that

$$\int_{t_0-\tau}^{t_0+\tau}\mathcal{H}^{n-2+a}(Z_t^{\tau,x_0,t_0})\,dt=0.$$

Hence we obtain $\mathcal{H}^{n-2+a}(Z_t)=0$ for a.e. $t\in[0,\infty)$. On the other hand, we compute that

$$\int_{t_0-\tau}^{t_0+\tau}\mu_t(Z_t^{\tau,x_0,t_0})\,dt\le\int_{t_0-\tau}^{t_0+\tau}\sum_{i=1}^{\infty}\sum_{t\in[t_i-r_i^2/\gamma_3^2,t_i+r_i^2/\gamma_3^2]}c_3D_1r_i^{n-1}\,dt$$
$$=\sum_{i=1}^{\infty}\int_{t_i-r_i^2/\gamma_3^2}^{t_i+r_i^2/\gamma_3^2}c_3D_1r_i^{n-1}\,dt=\sum_{i=1}^{\infty}\frac{2c_3D_1}{\gamma_3^2}r_i^{n+1}\le\frac{4c_3D_1}{\gamma_3^2\omega_n}\delta\mathcal{L}^n(B_1(x_0)).$$

Then $\delta'\to 0$ implies that $\int_{t_0-\tau}^{t_0+\tau}\mu_t(Z_t^{\tau,x_0,t_0})\,dt=0$. Thus we obtain $\mu(Z)=0$. $\square$

Lemma 5.7. For any $(y,s)\in\mathbb{R}^n\times[0,\infty)$ we have

$$\int_0^s\int_{\mathbb{R}^n}\frac{1}{2(s-t)}\rho_{y,s}(x,t)\,d|\xi_t^i|(x)dt\le c_4D_1\quad\text{for}\quad i\ge 1. \tag{5.20}$$

Proof. We compute that

$$\frac{d}{dt}\int_{\mathbb{R}^n}\rho_{y,s}(x,t)\,d\mu_t^i(x)\le\frac{1}{2(s-t)}\int_{\mathbb{R}^n}\rho_{y,s}(x,t)\,d\xi_t^i(x)=-\frac{1}{2(s-t)}\int_{\mathbb{R}^n}\rho_{y,s}(x,t)\,d|\xi_t^i|(x),$$

where Proposition 3.2 and (3.2) are used. Therefore we have

$$\int_0^s\frac{1}{2(s-t)}\int_{\mathbb{R}^n}\rho_{y,s}(x,t)\,d|\xi_t^i|(x)dt\le-\int_0^s\frac{d}{dt}\int_{\mathbb{R}^n}\rho_{y,s}(x,t)\,d\mu_t^i(x)dt$$
$$=\int_{\mathbb{R}^n}\rho_{y,s}(x,0)\,d\mu_0^i(x)-\lim_{t\uparrow s}\int_{\mathbb{R}^n}\rho_{y,s}(x,t)\,d\mu_t^i(x)\le\int_{\mathbb{R}^n}\rho_{y,s}(x,0)\,d\mu_0^i(x)\le c_4D_1,$$

where (3.12) is used. Hence we obtain (5.20). $\square$

We may assume that there exists a Radon measure ξ_t such that $\xi_t^i\to\xi_t$ as Radon measures. Define $d\xi:=d\xi_tdt$. Next we prove the vanishing of the discrepancy measure ξ.

Lemma 5.8. Assume that $\varphi^{\varepsilon_i,\delta_j}$ and $\mu_t^{\varepsilon_i,\delta_j}$ satisfy all the assumptions of Section 2 and $\mu_t^i=\mu_t^{\varepsilon_i,\delta_i}\to\mu_t$ and $\xi_t^i=\xi_t^{\varepsilon_i,\delta_i}\to\xi_t$ for any $t\ge 0$ as Radon measures. Then $\xi=0$.

Proof. By (5.20) and $\xi_t^i\to\xi_t$ we have

$$\int_{\mathbb{R}^n\times(0,s)}\frac{1}{2(s-t)}\rho_{y,s}(x,t)\,d|\xi|(x,t)\le c_4D_1.$$

Let R and T be positive numbers. We integrate with the measure $d\mu_sds$

$$\int_{B_R(0)\times[0,T+1]}\int_{\mathbb{R}^n\times(0,s)}\frac{1}{2(s-t)}\rho_{y,s}(x,t)\,d|\xi|(x,t)d\mu_s(y)ds$$
$$\le\int_{B_R(0)\times[0,T+1]}c_4D_1d\mu_s(y)ds\le c_3c_4D_1^2(T+1)R^{n-1}<\infty,$$

where (3.10) is used. By Fubini's theorem we obtain

$$\int_{\mathbb{R}^n\times[0,T+1]}\Big(\int_t^{T+1}\frac{1}{2(s-t)}\int_{B_R(0)}\rho_{(y,s)}(x,t)\,d\mu_s(y)ds\Big)d|\xi|(x,t)$$
$$\le c_3c_4D_1^2(T+1)R^{n-1}.$$

Hence there exists $c_7 = c_7(x,t) < \infty$ such that

$$\int_t^{t+1} \frac{1}{2(s-t)} \int_{B_R(0)} \rho_{(y,s)}(x,t)\, d\mu_s(y) ds \le c_7(x,t) < \infty \tag{5.21}$$

for $|\xi|$-a.e. $(x,t) \in \mathbb{R}^n \times [0,T]$.

Let $x \in B_{\frac{R}{2}}(0)$ and $T > s > t > 0$. We compute

$$\int_{\mathbb{R}^n} \rho_{y,s}(x,t)\, d\mu_s(y) \le \int_{B_R(0)} \rho_{y,s}(x,t)\, d\mu_s(y) + 2^{n-1} e^{-\frac{3}{8}\frac{(R/2)^2}{2(s-t)}} D_1, \tag{5.22}$$

where (5.6) is used. By (5.21) and (5.22) we obtain

$$\begin{aligned}&\int_t^{t+1} \frac{1}{2(s-t)} \int_{\mathbb{R}^n} \rho_{y,s}(x,t)\, d\mu_s(y) ds \\ \le& c_7(x,t) + \int_t^{t+1} \frac{1}{2(s-t)} 2^{n-1} e^{-\frac{3}{64}\frac{R^2}{s-t}} D_1\, ds < \infty\end{aligned} \tag{5.23}$$

for $|\xi|$-a.e. $(x,t) \in B_{\frac{R}{2}} \times [0,T]$ and for any $R > 0$ and $T > 0$. Hence (5.23) holds for $|\xi|$-a.e. $(x,t) \in \mathbb{R}^n \times [0,\infty)$. Set

$$h(s) = h_{x,t}(s) := \int_{\mathbb{R}^n} \rho_{y,s}(x,t)\, d\mu_s(y), \quad (x,t) \in \mathbb{R}^n \times [0,\infty).$$

Next, we claim that

$$\lim_{s \to t} h_{x,t}(s) = 0 \qquad |\xi|\text{-a.e. } (x,t) \in \mathbb{R}^n \times [0,\infty). \tag{5.24}$$

Define

$$A = \left\{ (x,t) \in \mathbb{R}^n \times [0,\infty) \,\middle|\, \int_t^{t+1} \frac{1}{2(s-t)} \int_{\mathbb{R}^n} \rho_{y,s}(x,t)\, d\mu_s(y) ds < \infty \right\}.$$

Note that $|\xi|(A^c) = 0$. Fix $(x,t) \in A$ and set $\lambda := \log(s-t)$. Then we have

$$\int_{-\infty}^0 h(t+e^\lambda)\, d\lambda = \int_t^{t+1} \frac{1}{s-t} \int_{\mathbb{R}^n} \rho_{y,s}(x,t)\, d\mu_s(y) ds < \infty. \tag{5.25}$$

Set $\kappa \in (0,1]$. By (5.25) there exists a sequence $\{\lambda_i\}_{i=1}^\infty$ such that

$$\lambda_i \downarrow -\infty, \qquad \lambda_i - \lambda_{i+1} \le \kappa, \qquad h(t+e^{\lambda_i}) \le \kappa. \tag{5.26}$$

Fix $\lambda \in (-\infty, \lambda_1]$ and choose i such that $\lambda \in [\lambda_i, \lambda_{i-1})$. Then by (3.9) we have

$$\begin{aligned} h(t+e^\lambda) &= \int \rho_{y,t+e^\lambda}(x,t)\, d\mu_{t+e^\lambda}(y) = \int \rho_{y,t+2e^\lambda}(x,t+e^\lambda)\, d\mu_{t+e^\lambda}(y) \\ &\le \int \rho_{y,t+2e^\lambda}(x,t+e^{\lambda_i})\, d\mu_{t+e^{\lambda_i}}(y) = \int \rho_x^R\, d\mu_{t+e^{\lambda_i}}, \end{aligned} \tag{5.27}$$

where $\frac{R^2}{2} = 2e^\lambda - e^{\lambda_i}$. On the other hand, by (5.26) we have

$$\kappa \ge h(t+e^{\lambda_i}) = \int \rho_{y,t+e^{\lambda_i}}(x,t)\, d\mu_{t+e^{\lambda_i}}(y) = \int \rho_x^r\, d\mu_{t+e^{\lambda_i}}, \tag{5.28}$$

where $\frac{r^2}{2} = e^{\lambda_i}$. Remark that there exists $c_8 > 0$ such that

$$1 \le \frac{R}{r} = \sqrt{2e^{\lambda-\lambda_i} - 1} \le 1 + c_8\kappa.$$

Let $1 > a > 0$ and $\kappa = \min\{a, \gamma_2(a)/c_8\}$ where $\gamma_2 = \gamma_2(a)$ is defined by Lemma 5.3. Then for $\lambda \le \lambda_1(a)$,

$$\begin{aligned} h(t+e^{\lambda}) \le & \int \rho_x^R \, d\mu_{t+e^{\lambda_i}} \\ \le & (1+a)\int \rho_x^r \, d\mu_{t+e^{\lambda_i}} + aD_1 \le 2\kappa + aD_1, \end{aligned}$$

where (5.27) and (5.28) are used. Passing $a \to 0$ we obtain

$$\lim_{s \downarrow t} h_{x,t}(s) = 0 \qquad \text{for } (x,t) \in A. \tag{5.29}$$

Thus we have (5.24). On the other hand, by Lemma 5.6 we obtain

$$\limsup_{s \downarrow t} h_{x,t}(s) \ge \frac{\eta}{2} > 0 \qquad \text{for } \mu\text{-a.e. } (x,t). \tag{5.30}$$

Hence by (5.24) and (5.30) we obtain

$$0 \ge \limsup_{s \downarrow t} h_{x,t}(s) \ge \frac{\eta}{2} \qquad \text{for } |\xi|\text{-a.e.},$$

where $|\xi| \ll \mu$ are used. Thus we have $|\xi| = 0$. $\square$

6. Proof of main results

Let $\{\varepsilon_i\}_{i=1}^\infty$ and $\{\delta_i\}_{i=1}^\infty$ be positive sequences with $\varepsilon_i, \delta_i \downarrow 0$. Set $\tilde\varphi^i \in C^{2,\alpha}_{loc}(\mathbb{R}^n)$ for $i \in \mathbb{N}$. Define measures $\tilde\mu^i$, $\tilde\xi^i$ and $\tilde V^i$ by

$$\tilde\mu^i(\phi) := \int_{\mathbb{R}^n} \phi\Big(\frac{\varepsilon_i |\nabla\tilde\varphi^i|^2}{2} + \frac{F_{\delta_i}(\tilde\varphi^i)}{\varepsilon_i}\Big)\, dx \quad \text{and} \quad \tilde\xi^i(\phi) := \int_{\mathbb{R}^n} \phi\Big(\frac{\varepsilon_i |\nabla\tilde\varphi^i|^2}{2} - \frac{F_{\delta_i}(\tilde\varphi^i)}{\varepsilon_i}\Big)\, dx$$

for $\phi \in C_c(\mathbb{R}^n)$, and

$$\tilde V^i(\psi) := \int_{\{x \,|\, |\nabla\tilde\varphi^i(x)| \ne 0\}} \psi(x, I - \nu^i \otimes \nu^i)\Big(\frac{\varepsilon_i |\nabla\tilde\varphi^i|^2}{2} + \frac{F_{\delta_i}(\tilde\varphi^i)}{\varepsilon_i}\Big)\, dx$$

for $\psi \in C_c(\mathbb{R}^n \times G_{n-1}(\mathbb{R}^n))$, where $\nu^i := \frac{\nabla\tilde\varphi^i}{|\nabla\tilde\varphi^i|}$. Note that $\tilde V^i \in \mathbf{V}_{n-1}(\mathbb{R}^n)$ and $\|\tilde V^i\| = \tilde\mu^i$. For $\phi \in C_c^2(\mathbb{R}^n)$, define

$$\mathcal{B}^i(\tilde\varphi^i, \phi) := \int_{\mathbb{R}^n} -\varepsilon_i \phi\Big(-\Delta\tilde\varphi^i + \frac{F'_{\delta_i}(\tilde\varphi^i)}{\varepsilon_i^2}\Big)^2 + \varepsilon_i \nabla\phi \cdot \nabla\tilde\varphi^i\Big(-\Delta\tilde\varphi^i + \frac{F'_{\delta_i}(\tilde\varphi^i)}{\varepsilon_i^2}\Big)\, dx.$$

The following lemma is obtained in the same manner as Lemma 9.3 of [18]. So we omit the proof.

Lemma 6.1. For $\phi \in C_c^2(\mathbb{R}^n)$ we assume that

(1) $\tilde\mu^i \to \tilde\mu$ as Radon measures on $\mathbb{R}^n$,
(2) $\tilde\xi^i$ is non-positive measure for $i \in \mathbb{N}$,
(3) $|\tilde\xi^i|\lfloor\{\phi > 0\} \to 0$ as Radon measures on $\mathbb{R}^n$,
(4) there exists $C > 0$ such that $\mathcal{B}^i(\tilde\varphi^i, \phi) \ge -C$ for $i \in \mathbb{N}$,
(5) $\mathcal{H}^{n-1}(\operatorname{spt}\tilde\mu \cap \{\phi > 0\}) < \infty$.

Then the following hold:

(1) $\tilde\mu\lfloor\{\phi > 0\}$ is $(n-1)$-rectifiable.
(2) There exists $\tilde V \in \mathbf{V}_{n-1}(\mathbb{R}^n)$ such that $\tilde V^i\lfloor\{\phi > 0\} \to \tilde V$ and $\|\tilde V\| = \tilde\mu\lfloor\{\phi > 0\}$.

(3) For any $Y \in C_c^1(\{\phi > 0\}; \mathbb{R}^n)$ we have

$$\delta \tilde{V}(Y) = \lim_{i\to\infty} \int -\varepsilon_i Y \cdot \nabla \tilde{\varphi}^i \Big(-\Delta \tilde{\varphi}^i + \frac{F'_{\delta_i}(\tilde{\varphi}^i)}{\varepsilon_i^2} \Big)\, dx. \tag{6.1}$$

(4) There exists the generalized mean curvature vector H for $\tilde{V}$ with

$$\int_{\mathbb{R}^n} \psi |H|^2 \, d\tilde{\mu} \leq \frac{2}{\pi} \liminf_{i\to\infty} \int_{\mathbb{R}^n} \varepsilon_i \psi \Big(-\Delta \tilde{\varphi}^i + \frac{F'_{\delta_i}(\tilde{\varphi}^i)}{\varepsilon_i^2} \Big)^2 dx < \infty \tag{6.2}$$

for $\psi \in C_c^2(\{\phi > 0\}; \mathbb{R}^+)$.

(5)

$$\limsup_{i\to\infty} \mathcal{B}^i(\tilde{\varphi}^i, \phi) \leq \mathcal{B}(\tilde{\mu}, \phi). \tag{6.3}$$

Proof of Theorem 2.7

First we prove Brakke's inequality. Let $\varphi^{\varepsilon_i, \delta_j} \in C^{2,\alpha}_{loc}(\mathbb{R}^n \times (0,\infty))$ be as in Theorem 2.7. Then by Proposition 4.3 there exist subsequences $\{\varepsilon_{i_k}\}_{k=1}^\infty, \{\delta_{j_k}\}_{k=1}^\infty$ and $\{\mu_t\}_{t\geq 0}$ such that (2.22), (2.23) and (2.24) hold. By Lemma 5.8, there exist subsequences $\{\varepsilon_{i_k}\}_{k=1}^\infty$ and $\{\delta_{j_k}\}_{k=1}^\infty$ (denoted by the same index) such that

$$\xi^{\varepsilon_{i_k}, \delta_{j_k}} \to 0 \quad \text{as Radon measures on} \quad \mathbb{R}^n \times [0,\infty). \tag{6.4}$$

Set $\varphi^k = \varphi^{\varepsilon_{i_k}, \delta_{j_k}}$, $\mu_t^k = \mu_t^{\varepsilon_{i_k}, \delta_{j_k}}$, $\xi_t^k = \xi_t^{\varepsilon_{i_k}, \delta_{j_k}}$ and $\xi^k = \xi^{\varepsilon_{i_k}, \delta_{j_k}}$. Let $t_0 \geq 0$ and $\phi \in C_c^2(\mathbb{R}^n; \mathbb{R}^+)$. If $\overline{D}_t \mu_t(\phi)\Big|_{t=t_0} = -\infty$, then (2.2) holds. Therefore we assume that

$$C_0 := \overline{D}_t \mu_t(\phi)\Big|_{t=t_0} > -\infty. \tag{6.5}$$

Then there exist $\{h_q\}_{q=1}^\infty$ and $\{t_q\}_{q=1}^\infty$ such that $h_q \downarrow 0, t_q \to t_0$ as $q \to \infty$ and

$$C_0 - h_q \leq \frac{\mu_{t_q}(\phi) - \mu_{t_0}(\phi)}{t_q - t_0} \quad \text{for} \quad q \geq 1.$$

We may assume that $t_q > t_0$ for any $q \geq 1$. (The other case is similar.)

By $\mu_t^k \to \mu_t$ and (6.4) there exists a subsequence $\{k_q\}_{q=1}^\infty$ such that

$$C_0 - 2h_q \leq \frac{\mu_{t_q}^{k_q}(\phi) - \mu_{t_0}^{k_q}(\phi)}{t_q - t_0} = \frac{1}{t_q - t_0} \int_{t_0}^{t_q} \frac{d}{dt} \mu_t^{k_q}(\phi)\, dt \tag{6.6}$$

and

$$\int_{\{\phi>0\}\times[t_0,t_q]} d|\xi^{k_q}| \leq h_q^2 (t_q - t_0) \tag{6.7}$$

for $q \geq 1$. By Proposition 3 and Lemma 4.1 there exists $C_1 = C_1(n, \phi, D_1) > 0$ such that

$$\frac{d}{dt} \mu_t^k(\phi) \leq C_1 \quad \text{for } k \geq 1,\ t \geq 0.$$

We may assume $C_1 > C_0$. Set

$$Z_q = \{ t \in [t_0, t_q] \mid \frac{d}{dt} \mu_t^{k_q}(\phi) \geq C_0 - 3h_q \} \quad \text{for } q \geq 1.$$

By (6.6) we have

$$C_0 - 2h_q \leq \frac{1}{t_q - t_0} \int_{[t_0,t_q]\setminus Z_q} C_0 - 3h_q \, dt + \frac{1}{t_q - t_0} \int_{Z_q} C_1 \, dt.$$

Hence we obtain

$$|Z_q| \geq \frac{(t_q - t_0)h_q}{C_1 - C_0 + 3h_q} \geq \frac{(t_q - t_0)h_q}{2(C_1 - C_0)} \tag{6.8}$$

for sufficiently large $q \geq 1$. By (6.7) we have

$$|Z_q| \inf_{t \in Z_q} |\xi_t^{k_q}|(\{\phi > 0\}) \leq h_q^2(t_q - t_0). \tag{6.9}$$

By (4.2), (6.8) and (6.9) for any $q \geq 1$ there exists $s_q \in Z_q$ such that

$$C_0 - 3h_q \leq \frac{d}{dt}\mu_t^{k_q}(\phi)\Big|_{t=s_q} = \mathcal{B}^{k_q}(\varphi^{k_q}(\cdot, s_q), \phi) \tag{6.10}$$

and

$$|\xi_{s_q}^{k_q}|(\{\phi > 0\}) \leq 3(C_1 - C_0)h_k. \tag{6.11}$$

Assume that the subsequence $\{\mu_{s_q}^{k_q}\}_{q=1}^\infty$ converges to a Radon measure $\tilde{\mu}$. By Lemma 4.1 and (6.5), it is possible to prove (see [19, 7.1]) that

$$\tilde{\mu}\lfloor\{\phi > 0\} = \mu_{t_0}\lfloor\{\phi > 0\}. \tag{6.12}$$

Hence, by Lemma 6.1, (5.15), (6.5), (6.10), (6.11) and (6.12) we have

$$\overline{D}_t\mu_t(\phi)\Big|_{t=t_0} \leq \mathcal{B}(\mu_{t_0}, \phi).$$

Therefore by this and Proposition 2.6 (2), $\{\mu_t\}_{t\geq 0}$ is a global solution for Brakke's mean curvature flow with initial data $\mu_0 = \frac{\pi}{2}\mathcal{H}^{n-1}\lfloor\Gamma_0$. Thus we obtain (a).

Next we prove (b). Set $w^k := \Phi_k \circ \varphi^k$, where $\Phi_k(s) := \sigma_{\delta_{j_k}}^{-1}\int_{-(1-\delta_{j_k})^{-1}}^s \sqrt{2F_{\delta_{j_k}}(y)}\, dy$ and $\varphi^k := \varphi^{\varepsilon_{i_k}, \delta_{j_k}}$. Note that $\Phi_k(-(1-\delta_{j_k})^{-1}) = 0$ and $\Phi_k((1-\delta_{j_k})^{-1}) = 1$. We denote $\varepsilon = \varepsilon_{i_k}$ and $\delta = \delta_{j_k}$. We compute that

$$|\nabla w^k| = \sigma_\delta^{-1}|\nabla\varphi^k|\sqrt{2F_\delta(\varphi^k)} \leq \sigma_\delta^{-1}\Big(\frac{\varepsilon|\nabla\varphi^k|^2}{2} + \frac{F_\delta(\varphi^k)}{\varepsilon}\Big).$$

Hence by (4.3) we have

$$\int_{\mathbb{R}^n} |\nabla w^k(\cdot, t)|\, dx \leq \int_{\mathbb{R}^n} \sigma_\delta^{-1}\Big(\frac{\varepsilon|\nabla\varphi^k|^2}{2} + \frac{F_\delta(\varphi^k)}{\varepsilon}\Big)\, dx \leq \sigma_\delta^{-1}D_1 \tag{6.13}$$

for $t \geq 0$. Fix $T > 0$. By the similar argument and (4.3) we obtain

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^n} |\partial_t w^k|\, dxdt &\leq \sigma_\delta^{-1}\int_0^T\int_{\mathbb{R}^n}\Big(\frac{\varepsilon|\partial_t\varphi^k|^2}{2} + \frac{F_\delta(\varphi^k)}{\varepsilon}\Big)\, dxdt \\ &\leq \frac{\sigma_\delta^{-1}}{2}\int_0^T\int_{\mathbb{R}^n}\varepsilon\Big(\Delta\varphi^k - \frac{F_\delta'(\varphi^k)}{\varepsilon^2}\Big)^2\, dxdt + \sigma_\delta^{-1}D_1T \leq \sigma_\delta^{-1}D_1(1+T).\end{aligned} \tag{6.14}$$

By (6.13) and (6.14), $\{w^k\}_{k=1}^\infty$ is bounded in $BV(\mathbb{R}^n \times [0, T])$. By the standard compactness theorem and the diagonal argument there is subsequence $\{w^k\}_{k=1}^\infty$ (denoted by the same index) and $w \in BV_{loc}(\mathbb{R}^n \times [0, \infty))$ such that

$$w^k \to w \quad \text{in } L^1_{loc}(\mathbb{R}^n \times [0, \infty)) \tag{6.15}$$

and a.e. pointwise. We denote $\varphi(x, t) := \lim_{k\to\infty}(1 + \Phi_k^{-1} \circ w^k(x, t))/2$. Then we have

$$\varphi^k \to 2\varphi - 1 \quad \text{in } L^1_{loc}(\mathbb{R}^n \times [0, \infty))$$

and a.e. pointwise. Hence we obtain (b1). By Proposition 2.6 (3) we obtain (b2). We have $\varphi^k \to \pm 1$ a.e. and $\varphi = 1$ or $= 0$ a.e. on $\mathbb{R}^n \times [0, \infty)$ by the boundedness of $\int_{\mathbb{R}^n} \frac{F_\delta(\varphi^k)}{\varepsilon}\, dx$.

Moreover $\varphi = w$ a.e. on $\mathbb{R}^n \times [0,\infty)$. Thus $\varphi \in BV_{loc}(\mathbb{R}^n \times [0,\infty))$. For any bounded open set $U \subset \mathbb{R}^n$ and a.e. $0 \le t_1 < t_2 < T$ we have

$$\begin{aligned}
&\int_U |\varphi(\cdot,t_2) - \varphi(\cdot,t_1)|\,dx = \lim_{k\to\infty} \int_U |w^k(\cdot,t_2) - w^k(\cdot,t_1)|\,dx \\
\le& \liminf_{k\to\infty} \int_U \int_{t_1}^{t_2} |\partial_t w^k|\,dtdx \le \liminf_{k\to\infty} \int_{\mathbb{R}^n}\int_{t_1}^{t_2} \Big(\frac{\varepsilon|\partial_t \varphi^k|^2}{2}\sqrt{t_2-t_1} + \frac{F_\delta(\varphi^k)}{\varepsilon\sqrt{t_2-t_1}}\Big)\,dtdx \\
\le& C_2 D_1 \sqrt{t_2-t_1},
\end{aligned} \tag{6.16}$$

where $C_2 = C_2(n,T) > 0$. By (6.16) and $|\Omega_0^+| < \infty$, $\varphi(\cdot,t) \in L^1(\mathbb{R}^n)$ for a.e. $t \ge 0$. By this and (6.16), we may define $\varphi(\cdot,t)$ for any $t \ge 0$ such that $\varphi \in C^{\frac{1}{2}}_{loc}([0,\infty); L^1(\mathbb{R}^n))$. Hence we obtain (b3). For $\phi \in C_c(\mathbb{R}^n;\mathbb{R}^+)$ and $t \ge 0$ we compute that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \phi\, d\|\nabla\varphi(\cdot,t)\| \le \liminf_{k\to\infty} \int_{\mathbb{R}^n} \phi|\nabla w^k|\,dx \\
\le& \lim_{k\to\infty} \sigma_{\delta_{j_k}}^{-1} \int_{\mathbb{R}^n} \phi\Big(\frac{\varepsilon_{i_k}|\nabla\varphi^k|^2}{2} + \frac{F_{\delta_{j_k}}(\varphi^k)}{\varepsilon_{i_k}}\Big)\,dx = \frac{2}{\pi}\int_{\mathbb{R}^n} \phi\, d\mu_t.
\end{aligned}$$

Hence we obtain (b4). □

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